# ETMAG

## LECTURE 15

- Systems of Equations - Cramer's Rule
- Inverse matrix
- Eigenvalues, eigenvectors

## Determinant and systems of linear equations

**Theorem.** (Uniqueness theorem)

A system of *n* linear equations with *n* unknowns

$$(*)\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \ ... + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \ ... + a_{2,n}x_n = b_2 \\ \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \ ... + a_{n,n}x_n = b_n \end{cases}$$

has a unique solution iff $\det(A) \neq 0$

**Proof.**

It follows from the fact that the corresponding homogeneous system has unique solution $\Theta$ iff rank($A$)=$n$ which, in turn is equivalent to $det(A) \neq 0$. Then, if (and that's a big IF) $v_0$ is a solution then all solutions $v$ of $(*)$ look like $v = \Theta + v_0 = v_0$.

**Warning.**

The uniqueness theorem is a "both ways" implication but is often misunderstood. The conclusion should be understood as "the set of solutions of $(*)$ has exactly one element". Hence the negation of this is (contrary to what many people believe) not

"if $\det(A) = 0$ then $(*)$ it is not true that $(*)$ has a solution"

but rather (remember de Morgan's Law!)

"if $\det(A) = 0$ then it is not true that the set of solutions of $(*)$ has exactly one element" (which means either none or more than one).

Look at this:

$\begin{cases} x + y = 2 \\ 2x + 2y = 4 \end{cases}$ $\det\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 4 - 4 = 0$ but the system has infinitely many solutions of the form $(t, 2 - t)$ where $t$ is any real number.

**Theorem.** (Cramer's Rule)
Let $A$ be an $n \times n$ matrix with $\det(A) \neq 0$ and let $B$ be any $n \times 1$ matrix. Then the system of equations $AX = B$ has unique solution $X = [x_1, x_2, \dots, x_n]^T$ and for each $i = 1, 2, \dots, n$

$$x_i = \frac{\det A_i}{\det A},$$

where $A_i$ is obtained by replacing the $i$-th column of $A$ with $B$.
**Proof** (skipped).

**Example.**

$$\begin{cases} 2x + 4y - z = 11 \\ -4x - 3y + 3z = -20 \\ 2x + 4y + 2z = 2 \end{cases} \quad |A| = \begin{vmatrix} 2 & 4 & -1 \\ -4 & -3 & 3 \\ 2 & 4 & 2 \end{vmatrix} = -12+16+24-6-24+32=30,$$

$$|A_1| = \begin{vmatrix} 11 & 4 & -1 \\ -20 & -3 & 3 \\ 2 & 4 & 2 \end{vmatrix} = -66+80+24-6-132+160=264-204=60,\ x=2$$

$$|A_2| = \begin{vmatrix} 2 & 11 & -1 \\ -4 & -20 & 3 \\ 2 & 2 & 2 \end{vmatrix} = -80+8+66-40-12+88 = 162-132 = 30,\ y=1$$

$$|A_3| = \begin{vmatrix} 2 & 4 & 11 \\ -4 & -3 & -20 \\ 2 & 4 & 2 \end{vmatrix} = -12-160-176+66+160+32 = 258-348 = -90,\ z=-3$$

**Definition.** (Inverse matrix)
Let $A$ be an $n \times n$ matrix. If there exists a matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$ then $A^{-1}$ is called *the inverse* (*matrix*) of $A$.

**Fact.** The inverse matrix for $A$, if it exists, is unique.

This follows from the very general fact the in every associative algebra the inverse element, if there is one, is unique.

**Theorem.**
A matrix is $A$ invertible iff $\det(A) \neq 0$.

**Proof.**($\Rightarrow$)
If $A^{-1}$ exists, then $\det(AA^{-1}) = \det(I) = 1 = \det(A)\det(A^{-1})$ hence both $\det(A)$ and $\det(A^{-1})$ are different from zero.

($\Leftarrow$)

If $\det(A) \neq 0$ then, from the uniqueness theorem for $n$×$n$ systems of equations, for every $n$×1 matrix $B$ there exists a (unique) solution of the system $AX = B$. Replacing $B$ with consecutive columns of the identity matrix $I$ we get the existence of the corresponding columns of the inverse matrix which in turn proves the existence of the inverse matrix itself.

To be more specific:

$$A^{-1} = X = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

Consider $\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{n,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. The system is uniquely solvable and the solution, $X_1$ is the first column of $A^{-1}$. The same can be said about the second, third and each next column of $X$ and $I$. QED

The proof suggests a method (two methods, really) for finding $A^{-1}$ (doing proofs makes sense):

Method 1.

Row-reduce the following matrix to a row-canonical one

$$[A|I] = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & 1 & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & 0 & 0 & \dots & 1 \end{bmatrix} \sim \dots \sim \dots \sim \dots \sim$$

$$\sim \dots \sim \begin{bmatrix} 1 & 0 & \dots & 0 & x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ 0 & 1 & \dots & 0 & x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{bmatrix} = [I|A^{-1}]$$

This is always possible if A is invertible. It proves that $A$ is invertible iff it may be row-reduced to the identity matrix $I$.

Method 2.

Using Cramer's rule to calculate each $x_{i,j}$ of $A^{-1}$.

This method involves calculation of det($A$) and $n^2$ determinants of the size $(n-1) \times (n-1)$. For large matrices it takes forever.

$x_{i,j}$ appears in $j$-th column of $A^{-1}$ which means must consider

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_{1,j} \\ \vdots \\ x_{i,j} \\ \vdots \\ x_{n,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = I_j$$ where the solitary 1 in $I_j$

is in the $j$-th position. So, in order to find the $i$-th unknown we need divide the determinant of $A^*_{i,j}$ ($A$ with $i$-th column replaced by $I_j$) by det($A$).

$$detA_{i,j}^{*} = det\begin{bmatrix} a_{1,1} & \cdots & 0 & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{j,1} & \cdots & \textcircled{1} & \cdots & a_{j,n} \\ \vdots & & \vdots & \cdots & \vdots \\ a_{n,1} & & 0 & \cdots & a_{n,n} \end{bmatrix}$$

in j-th row and i-th column of $A_{i,j}^{*}$

If you do this determinant by *i*-th column, the only nonzero term in the Laplace expansion will be $(-1)^{i+j}$ times the determinant obtained by the removal of *j*-th row and *i*-th column from $A_{i,j}^{*}$. Here is the funny thing: $A$ and $A_{i,j}^{*}$ only differ on the *i*-th column, which is being removed. Hence $detA_{i,j}^{*} = (-1)^{i+j}detA_{j,i}$ and, finally, $x_{i,j} = \frac{(-1)^{i+j}detA_{j,i}}{detA}$. In other words

$$A^{-1} = \frac{1}{detA}\left[(-1)^{i+j}detA_{i,j}\right]^{T}$$

**Example.**

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix}.$$ Find $A^{-1}$.

Method 1. (Gauss elimination). Notice and remember the strategy used:

Step one: get number 1 in the upper left corner

$$\begin{bmatrix} 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} r_4 - r_2 \\ r_3 - 2r_2 \\ r_1 \leftrightarrow r_2 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 & 0 & -2 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} r_2 + 2r_3 \\ r_1 + r_3 \\ r_4 + 2r_3 \end{matrix} \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -7 & 1 & 1 & -4 & 2 & 0 \\ 0 & -1 & -4 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \quad \begin{matrix} r_1 + 2r_4 \\ r_2 + 7r_4 \\ r_3 + 4r_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -6 & -11 & 2 & 7 \\ 0 & -1 & 0 & 0 & -4 & -6 & 1 & 4 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} r_3 \leftrightarrow r_2 \\ r_4 \leftrightarrow r'_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -3 & 1 & 2 \\ 0 & -1 & 0 & 0 & -4 & -6 & 1 & 4 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -6 & -11 & 2 & 7 \end{bmatrix}$$

$$\sim(-1)r_2 \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & -3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 4 & 6 & -1 & -4 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -6 & -11 & 2 & 7 \end{bmatrix}$$ so, $$A^{-1} = \begin{bmatrix} -2 & -3 & 1 & 2 \\ 4 & 6 & -1 & -4 \\ -1 & -1 & 0 & 1 \\ -6 & -11 & 2 & 7 \end{bmatrix}$$

Method 2. (Cramer's Rule, cofactors)

$\det A = 1$ (we are cheating, we derive this from method 1. Only the operation $(-1)r_2$ and three row-swaps in the previous slide affected the determinant. Scaling a row by (-1) and each row-swap changes the sign of the determinant.)

Let's calculate just a single entry of $A^{-1}$, say $A^{-1}(2,3)$. According to the cofactor theorem $A^{-1}(2,3) = \frac{1}{detA}(-1)^{2+3}\det(A_{3,2}) = -\det(A_{3,2})$.

$$\det(A_{3,2}) = \begin{vmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{vmatrix} = 4 - 2 - 1 = 1$$ which means, $A^{-1}[2,3]$ should be $-1$. We move back one slide and … surprise, surprise! it checks. And now you only have to calculate the remaining 15 entries of $A^{-1}$

**Example 2**. (Cramer's Rule, cofactors, 3×3 matrix)

$A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & -2 & -5 \\ -1 & 1 & 1 \end{bmatrix}$, find $A^{-1}$. First, we find |A| = $4 + 12 + 5 - 6 - 10 - 4 = 1$. Next transpose the matrix of cofactors

$$A^{-1} = \begin{bmatrix} \begin{vmatrix} -2 & -5 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & -5 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} & -\begin{vmatrix} -2 & 3 \\ 4 & -5 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^T =$$

$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$. Please, check this result by matrix multiplication.

# EIGENVALUES AND EIGENVECTORS

**Definition.**
Let $A$ be an $n \times n$ matrix over $\mathbb{K}$. Every scalar $\lambda$ such that for a nonzero vector $X_\lambda$ from $\mathbb{K}^n$, $AX_\lambda = \lambda X_\lambda$ is called an *eigenvalue* of $A$ and each $X_\lambda$ is called an *eigenvector* belonging to $\lambda$.

These things have a wide range of applications, from differential equations, through big data systems, through graph theory.

**Example.**

Find eigenvalues of $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix}$.

This is equivalent to discussing in terms of $\lambda$ the existence of nonzero vector [a,b,c] and such that

$$\begin{cases} 0a - b + 2c = \lambda a \\ -2a - b + 4c = \lambda b \\ -2a - 2b + 5c = \lambda c \end{cases}$$

. Moving right-hand side to the left we get

$$\begin{cases} -\lambda a - b + 2c = 0 \\ -2a + (-1-\lambda)b + 4c = 0 \\ -2a - 2b + (5-\lambda)c = 0 \end{cases}$$

. We are looking for nonzero solutions to this homogeneous system of equations. They exist iff the dimension of the solution space is nonzero, which means the rank of the coefficient matrix is less than 3, which in turn means the determinant of the matrix is 0.

$$\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ -2 & -2 & 5-\lambda \end{vmatrix} \underset{(r_3 - r_2)}{=}$$

$$\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ 0 & -1+\lambda & 1-\lambda \end{vmatrix} \underset{take\ out\ common\ factor\ from\ r_3}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & -1 & 2 \\ -2 & -1-\lambda & 4 \\ 0 & -1 & 1 \end{vmatrix} \underset{(c_2 + c_3)}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 & 2 \\ -2 & 3-\lambda & 4 \\ 0 & 0 & 1 \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} \underset{(r_2 - r_1)}{=}$$

$$(1-\lambda)\begin{vmatrix} -\lambda & 1 \\ \lambda-2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & 1 \end{vmatrix} =$$

$(1-\lambda)^2(2-\lambda) = 0$. Hence, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

This example can be easily generalized to

**Theorem.**
A scalar $\lambda$ is an eigenvalue for $A$ iff $\det(A - \lambda \mathrm{I}) = 0$.

**Proof.** Just as we did in the example, instead of $AX = \lambda X$ we can write $AX = (\lambda I)X$ which leads to $(A - \lambda I)X = \Theta$. Nonzero solutions to an $n \times n$ homogeneous system of equations exist iff the determinant of the coefficient matrix is zero.

**Fact.**
For every eigenvalue $\lambda$ of $A$ the set $W_\lambda = \{X \in \mathbb{K}^n | AX = \lambda X\}$ is a subspace in $\mathbb{K}^n$. The subspace is called an *eigenspace* for $\lambda$.

**Proof.** $W_\lambda$ is the solution space for $(A - \lambda I)X = \Theta$.

We are on familiar grounds now, we can solve homogeneous systems of equations. We have to do it separately for each eigenvalue, though.

**Example - continued.** Knowing that the eigenvalues are $\lambda_1$, $\lambda_2 = 1$ and $\lambda_3 = 2$ find eigenvectors of $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix}$ and the dimension of each eigenspace.

For $\lambda$=1 our system of equations is reduced to $\begin{bmatrix} -1 & -1 & 2 \\ -2 & -2 & 4 \\ -2 & -2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $r_3 = r_2$ and $r_2 = 2r_1$ the rank of the matrix system is 1 and the system is equivalent to $-a - b + 2c = 0$ which means $a$=$-b + 2c$ and $b$,$c$ run free. So all eigenvectors for $\lambda$=1 look like $(-b + 2c, b, c)$ = $b(-1,1,0) + c(2,0,1)$ so $\dim\left(W_{\lambda_1}\right) = 2$.

For $\lambda$=2 we get $\begin{bmatrix} -2 & -1 & 2 \\ -2 & -3 & 4 \\ -2 & -2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Row reducing the matrix we get

$$\begin{bmatrix}-2 & -1 & 2\\ -2 & -3 & 4\\ -2 & -2 & 3\end{bmatrix}\underset{r_2-r_1,\, r_3-r_1}{\sim}\begin{bmatrix}-2 & -1 & 2\\ 0 & -2 & 2\\ 0 & -1 & 1\end{bmatrix}\underset{r_1-r_3,\, r_2-2r_3}{\sim}$$

$\begin{bmatrix}-2 & 0 & 1\\ 0 & 0 & 0\\ 0 & -1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$. The rank is 2, dimension of the solution space is 1. We get $a-\frac{1}{2}c=0$ and $b$-$c$=0, i.e. $a=\frac{1}{2}c$ and $b=c$ and eigenvectors are $(\frac{1}{2}c, c, c)$, for all $c\neq 0$, or, for example, $t(1,2,2)$ for all nonzero $t\in\mathbb{R}$. Hence $\dim\left(W_{\lambda_2}\right)=1$.

**Definition.**
An $n \times n$ matrix A is said to be *similar* to B iff there exists a matrix $P$ such that $A = P^{-1}BP$. We write $A \approx B$.

Note that *similarity* should not be confused with *row equivalence*.

**Fact.**
Similarity of matrices is an equivalence relation on the set $\mathbb{K}^{n \times n}$.

**Fact.**
Suppose $A \approx B$ with $A = P^{-1}BP$. Then

*1.* $\det(A) = \det(B)$ (*obvious*)

2. for every $n \in \mathbb{N}, A^n = P^{-1}B^nP$ (*obvious*)

*3.* $\det(A - \lambda I) = \det(B - \lambda I)$.

*4.* $A$ and $B$ have the same eigenvalues. (*consequence of 3.*)

**Proof** (3). $A - \lambda I = P^{-1}BP - \lambda I = P^{-1}BP - \lambda P^{-1}IP = P^{-1}(B - \lambda I)$P hence $A - \lambda I \approx B - \lambda I$ so, by *1.*, their determinants are equal.

**Theorem.**

An $n \times n$ matrix A is similar to a diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

iff the diagonal entries $a_1, a_2, \dots, a_n$ are eigenvalues of A and there exists a basis R=$\{v_1, v_2, \dots, v_n\}$ for $\mathbb{K}^n$ such that for each $i$=1,2,…,$n$ $v_i$ is an eigenvalue belonging to $a_i$. In such a case, if we write $D = P^{-1}AP$ then columns of $P$ are (vertical) vectors $v_1, v_2, \dots, v_n$. This is to say that if we decide to express similarity of the two matrices in the form $A = R^{-1}DR$ then $R^{-1}$ has vectors $v_1, v_2, \dots, v_n$ as columns.

**Example - continued.**

The last theorem says that our matrix from the last example,

$$A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix}$$ is similar to $$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ and the

change-of-basis $$P = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} -1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 4 \end{matrix}$$

$$A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & -1 & 4 \\ -2 & -2 & 5 \end{bmatrix} \begin{matrix} -1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 4 \end{matrix}$$

Checks!